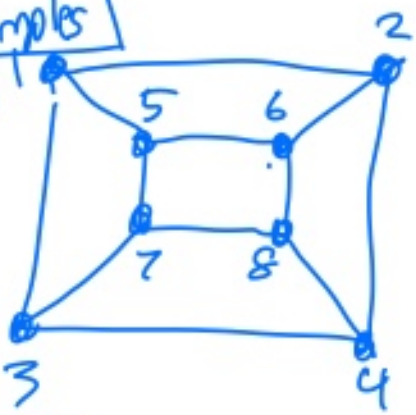
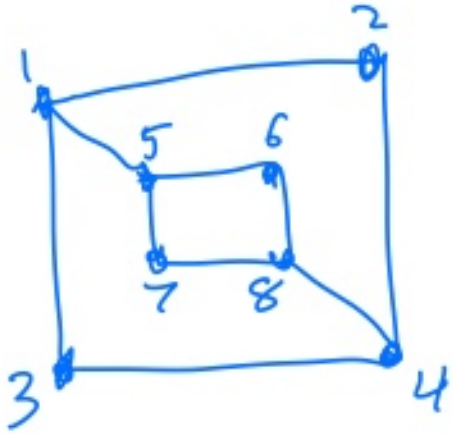
Given a graph (multigraphs, loops OK), a Hamilton (or Hamiltonian) path is a path that visits every vertex exactly once. (If the graph is a circuit, it's called a Hamiltonian circuit.)

Question: For which graphs can I find a Hamiltonian path or circuit?

Examples

① 

② 

Answers:

① $2 \to 1 \to 3 \to 4 \to 8 \to 7 \to 5 \to 6 \to 2$
circuit!

or $1 \to 2 \to 4 \to 3 \to 7 \to 8 \to 6 \to 5 \to 1$

or $1 \to 5 \to 7 \to 3 \to 4 \to 8 \to 6 \to 2 \to 1$

Yes, we can find it!

② $7 \to 5 \to 6 \to 8 \to 4 \to 2 \to 1 \to 3$

Yes, we can find it!

or

③ 

impossible!

There is no known simple algorithm for detecting if a graph has a Hamiltonian circuit or path. [And, also, no nice algorithm to find the path/circuit.].

Dirac's Graph Theorem – ~~If every vertex of a graph has degree at least 3, then there exists a Hamiltonian circuit on this graph.~~ connected/undirected

Corrected 9/13/24

If a simple graph has $n \geq 3$ vertices and the degree of each vertex is at least $\frac{n}{2}$, then there exists a Hamiltonian circuit on the graph.

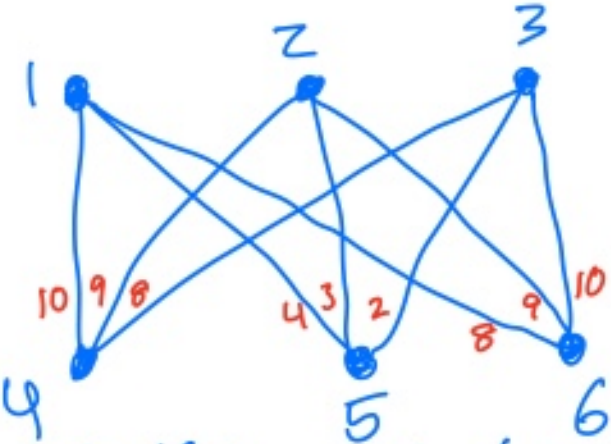
Weighted graphs – graphs for which each edge has a weight (we can pretend the weights are lengths of edges).

weight ↑ a real #.

Question: Given a weighted graph (that is simple and connected & undirected), given two vertices of the graph, can we find a shortest distance path that goes between the two vertices? (Using a simple algorithm)

Example

$K_{3,3}$



What is the shortest path from vertex 4 to vertex 6?

Shortest has length 18: possible paths $4 \to 1 \to 6$, $4 \to 3 \to 6$, $4 \to 2 \to 6$

Not 4→2→5→7→6.

Dijkstra's Algorithm – "a greedy" algorithm.

Given a graph, you wish to go from vertex a to vertex z. Let {b, c, ..., d} be all vertices adjacent to a. Find the shortest path to one of them — i.e. the edge with the smallest weight — say it's b. Now consider those to be a set {a, b, ..., d} — Next find all the vertices that are one edge away from that set. Then find the shortest path from the first vertex to the edge of that set: call the new set {a, e, ..., f}. Update your path to that length. Keep going until you hit the final vertex, and you will have the shortest path to z.

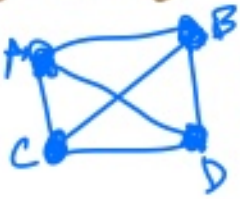
call the next set {a, b, c, ...}

Changing the question a little bit:

Traveling Salesman Problem: Find a Hamiltonian circuit for the graph that has the least length.

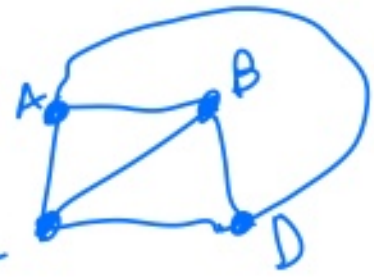
↑ unsolved problem to find a polynomial time algorithm for this.

Planar Graphs – Types of connected, undirected graphs that can be embedded in a flat plane without crossings of edges.

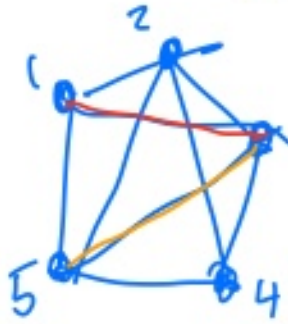
Examples ① 

is planar, since it is

the same as



(2)



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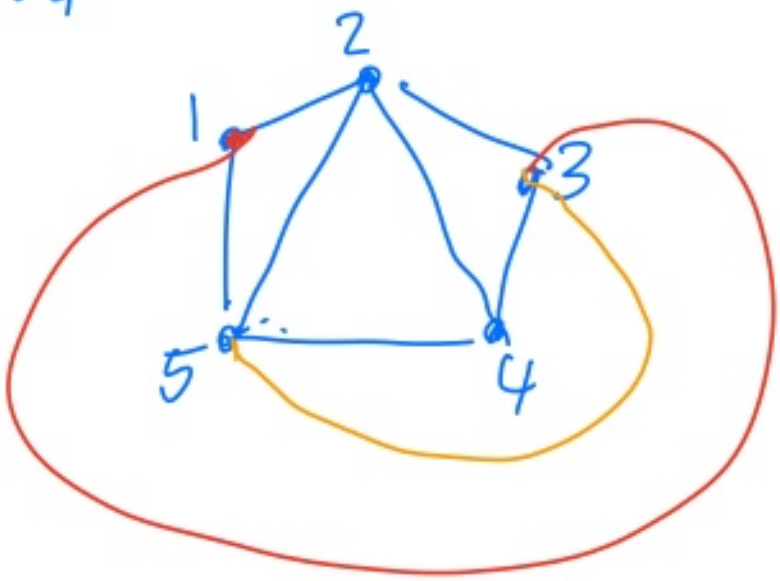
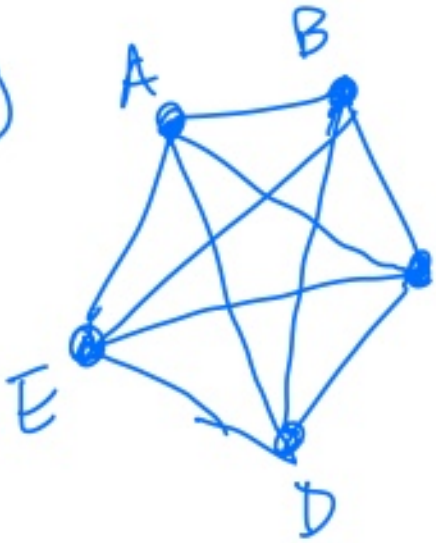


(3)

K_5



seems impossible!

(4)